Skyrmions in nematic liquid crystals

A. N. Bogdanov* and U. K. Rößler†
Leibniz-Institut für Festkörper- und Werkstoffforschung Dresden, Postfach 270116, D-01171 Dresden, Germany

A. A. Shestakov
Donetsk Institute for Physics and Technology, 340114 Donetsk, Ukraine
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Analytical solutions for static two-dimensional axisymmetric localized states minimizing the Frank free energy for nematic liquid crystals have been derived. These solitonic structures (*skyrmions*) include the well-known Belavin-Polyakov solutions as a special case for equal elastic constants. The structure and the equilibrium parameters of these nematic skyrmions crucially depend on values of the elastic constants. Stability limits of these structures and the possibility to observe them in nematic liquid crystals are discussed.

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I. INTRODUCTION

Inhomogeneous localized structures are a subject of intensive investigations in many nonlinear field models of modern physics [1,2]. In this paper we consider a special type of localized configurations in a classical three-dimensional vector field **n** (Fig. 1), where **n** has three components and $|\mathbf{n}|=1$. This texture is homogeneous along a certain direction (z axis in this paper); the vector **n** is parallel to the z axis at the center $[\mathbf{n}=(0,0,1)]$ and, by continuous rotation along radial directions in the *xoy* basal plane, it approaches the antiparallel orientation $[\mathbf{n}=(0,0,-1)]$ when the distance from the center tends to infinity (Fig. 1).

Mathematically these two-dimensional nonsingular localized structures represent *mappings* of the *xoy* basal plane, i.e., R^2 onto the unit sphere S^2 of *spin states* (directions of the vector **n**). Because the vector **n** tends to the same value $[\mathbf{n}=(0,0,-1)]$ for all points at infinity the plane *xoy* may be folded into a spherical surface [1]. By identification of points at infinity in R^2, the space R^2 is compactified into S^2, and the vector configurations $\mathbf{n}(x,y)$ in our localized structures are recognized as mappings between unit spheres $S^2 \to S^2$. Thus, they may be classified by the homotopy group $\Pi_2(S^2)$ [1,3]. In general, the configurations $\mathbf{n}(x,y)$ in these patterns need not have any symmetry within the *xoy* basal plane. However, in known physical models the internal symmetry of the system imposes the axial symmetry on the possible localized solutions. Thus, in this paper we consider only axially symmetric configurations of $\mathbf{n}(x,y)$ as simplest versions of localized structures. Examples are displayed in Fig. 1.

As physical objects, these structures first were described for a classical model of an isotropic ferromagnet in Ref. [4]. In that paper, Belavin and Polyakov demonstrated that the localized states (Fig. 1) belong to metastable states of the ferromagnet and derived analytical solutions for $\mathbf{n}(x,y)$ describing them. The idea that such topologically nontrivial textures may arise in real physical systems had a strong impact on modern physics. Such structures (named *skyrmions* by analogy with localized structures in the Skyrme model for mesons and baryons [5]) became an object of intensive investigations in many condensed-matter models. Skyrmions are believed to play an important role in dynamics of magnetic systems [6] and can be stabilized by surface-induced interactions in magnetic nanostructures [7]. Skyrmion configurations play an important role in quantum Hall systems [8], Bose-Einstein condensates [9], and magnetic semiconductors [10]. It was also found that the formation of skyrmions accompanies the introduction of holes in the CuO_2 plane of high-T_c superconductors [11]. Beyond models in condensed-matter physics, Belavin-Polyakov skyrmions are central objects of interest investigated in modern field theory [12], and mathematical physics [13].

In the course of various investigations about these localized structures in different fields of nonlinear physics, various different names have been coined. The term ‘‘two-dimensional topological solitons’’ has been used in some magnetic models to designate topological features of these states, the terms ‘‘vortices’’ and ‘‘flux lines’’ are due to some physical analogy between textures in Fig. 1 and vortex states in the mixed state of superconductors [14] and in liquid helium [15]. Terms like ‘‘threads’’ or ‘‘filaments’’ are inspired by the resemblance with certain liquid crystal textures [16]. Finally, ‘‘skyrmion’’ and ‘‘baby skyrmion’’ imply their relation to the localized solutions earlier derived by Skyrme within his model for low-energy dynamics of mesons and baryons [5,17]. In fact, the Skyrme model [5] comprises three spatial dimensions, and the name ‘‘baby skyrmion’’ was used by some field theorists [17] to distinguish two-dimensional localized states from ‘‘mature’’ three-dimensional solutions in the original Skyrme model [5]. In recent years, the term ‘‘skyrmion’’ became generally accepted for such nonsingular two-dimensional localized states.

In this contribution, we investigate skyrmion structures for a general elasticity model of an orientable continuous medium, i.e., the Frank free energy for nematic liquid crys-

*Permanent address: Donetsk Institute for Physics and Technology, 340114 Donetsk, Ukraine. Electronic address: bogdanov@kinetic.ac.donetsk.ua

†Corresponding author: IFW Dresden, Postfach 270116, D-01171 Dresden, Germany. FAX: +49-351-4659-537. Electronic address: u.roessler@ifw-dresden.de

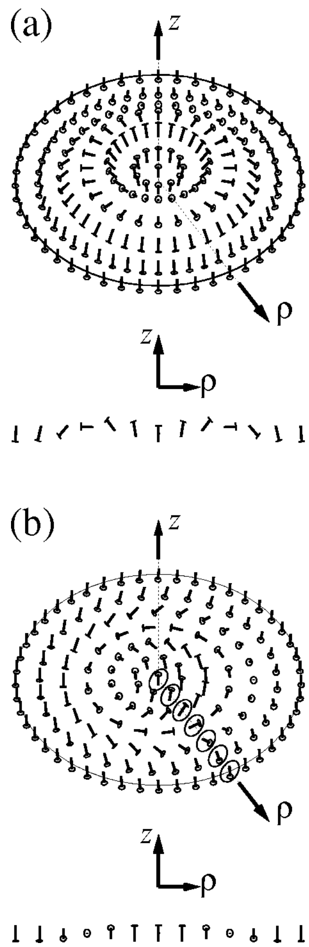

FIG. 1. Structure of the director field of axisymmetric nematic skyrmions in the *xoy* basal plane. Along the *z* direction, the configuration is homogeneous, and the center line of the skyrmions is along the *z* axis. In a skyrmion with phase angle $\alpha=0$ (a) the director **n** rotates in the ρoz plane. For phase angle $\alpha=\pi/2$ (b) **n** rotates in the plane perpendicular to the radial directions as indicated along the ρ vector. Lower panels show the directors in the *xoy* plane along the radial directions through the skyrmion center.

tals [16]. We show that special types of skyrmions can exist in nematic liquid crystals, and we derive analytical solutions for these nematic skyrmions by minimizing the Frank free energy. Further, we discuss their structure and stability limits. To motivate this work, we briefly anticipate a central result that will be derived below. The Frank energy contains the manifold of Belavin-Polyakov solutions [4] only in a special case within the so-called single-constant approximation. In the general case of an arbitrary set of elastic constants, this classical field-model puts restrictions on possible skyrmion structures. But, these restrictions do not fully lift the degeneracy of the corresponding solutions for *nematic skyrmions*. In a forthcoming companion paper, we will show that, in *cholesteric* liquid crystals, the degeneracy of solutions can be lifted to yield unique and stable skyrmions [18]. This is achieved by inclusion of particular terms in the Frank energy to describe the tendency for twisted configurations in cholesterics. Here, we restrict ourselves to the case of skyrmions in nematics to exemplify the restrictions for localized nonsingular structures in generalized elasticity models.

II. MODEL

Within the continuum theory of liquid crystals [16] the elastic energy (Frank energy) of a nematic system is written as

$$W=\int w\,d\mathbf{x}=\int \frac{1}{2}[K_1(\operatorname{div}\mathbf{n})^2+K_2(\mathbf{n}\cdot\operatorname{curl}\mathbf{n})^2 +K_3(\mathbf{n}\times\operatorname{curl}\mathbf{n})^2]\,d\mathbf{x}, \qquad (1)$$

where the director **n** is a unity vector, K_i are elastic constants associated with splay (K_1), twist (K_2), and bend (K_3) distortions [16]. In external electric (magnetic) fields dielectric (diamagnetic) response of the liquid crystal is described by additional energy density contributions

$$w_f=-\Delta\epsilon(\mathbf{n}\cdot\mathbf{E})^2-\Delta\chi(\mathbf{n}\cdot\mathbf{H})^2, \qquad (2)$$

where $\Delta\epsilon$ and $\Delta\chi$ are dielectric and diamagnetic anisotropies, respectively [16]. The equilibrium distributions of the vector field $\mathbf{n}(\mathbf{r})$ are determined by minimizing the Frank free energy (1) with the corresponding boundary conditions.

In this paper we consider skyrmions of the type shown in Fig. 1, i.e., axisymmetric structures with boundary conditions $\mathbf{n}(0)=(0,0,1)$, and $\mathbf{n}(\mathbf{r})=(0,0,-1)$ for $|\mathbf{r}|\to\infty$. These solutions may conveniently be expressed by using spherical coordinates for the vector **n** and cylindrical coordinates for the spatial variables

$$\mathbf{n}=[\sin\theta\cos\psi,\sin\theta\sin\psi,\cos\theta],\quad \mathbf{r}=(\rho,\phi,z). \qquad (3)$$

The skyrmions are assumed to be homogeneous along their axes taken as the *z* axis and, thus, the director **n** varies only in the basal plane, i.e., it depends only on ρ and ϕ. Moreover, rotational symmetry of these structures means that the solutions have the form $\theta(\rho),\psi(\phi)$. First, we consider the important special case of equal elastic constants ($K_1=K_2=K_3=K$). In this *single-constant approximation* the Frank energy (1) reduces to the functional form

$$W=\frac{K}{2}\int\sum_{i,j}\left(\frac{\partial n_i}{\partial x_j}\frac{\partial n_i}{\partial x_j}\right)d\mathbf{x} \qquad (4)$$

similar to the energy of an isotropic ferromagnet [16]. In particular, for functions $\theta(\rho)$, $\psi(\phi)$ the energy functional (1), in this case, can be written as

$$W=\int_0^{2\pi}d\phi\int_0^\infty\frac{K}{2}\left[\left(\frac{d\theta}{d\rho}\right)^2+\frac{\sin^2\theta}{\rho^2}\left(\frac{d\psi}{d\phi}\right)^2\right]\rho\,d\rho. \qquad (5)$$

The Euler equations for the functional (5) with the boundary conditions

$$\theta(0)=0,\quad \theta(\infty)=\pi \qquad (6)$$

have analytical solutions (Belavin and Polyakov [4]):

$$\theta=2\arctan(\rho/\rho_0)^N,\qquad \psi=N\,\phi+\alpha \qquad (7)$$

with the topological charge [1]

$$Q=\frac{1}{4\pi}\epsilon^{ijk}\int n_i\frac{\partial n_j}{\partial x}\frac{\partial n_k}{\partial y}dx\,dy=N, \tag{8}$$

where ϵ^{ijk} is the totally antisymmetric tensor and N is a nonzero integer that yields the number of windings for the vector $\mathbf{n}$ in the basal plane. The angle α varying in the interval $[0,\pi]$ and the parameter ρ_0 within $[0,\infty]$ describe the manifold of solutions (7). In particular, the localized structures in Fig. 1 represent skyrmions with the topological charge $Q=1$ and $\alpha=0$ (a) or $\alpha=\pi/2$ (b). Integration of Eqs. (5) with (7) yields the value $E_0=4\pi NK$ for the energy of the Belavin-Polyakov skyrmion. This energy does not depend on the parameters α and ρ_0. This remarkable degeneracy of the solutions is a consequence of the invariance of the skyrmion energy (7) under scale transformation of the profiles $\theta(\rho)\rightarrow\theta(\rho/\lambda)$ with $\lambda>0$ and under transformations $\psi\rightarrow\psi+\text{const}$.

Mathematically the functional Eq. (4) is related to so-called O(3) nonlinear sigma models (or CP_1 model) in general field theories [1], and it plays an important role in condensed-matter phenomenological models. That is why Belavin-Polyakov solutions found applications in theories for many physical systems and skyrmions have become important objects of interest in various modern theories (see references in the Introduction I). In the following section, we show how the solutions (7) behave in the general case with arbitrary values for the elastic constants in the Frank functional Eq. (1) for nematics.

III. SOLUTIONS FOR NEMATIC SKYRMIONS

For director configurations of type $\theta(\rho)$, $\psi(\phi)$ in nematics with nonequal elastic constants K_i the energy density in Eq. (1) has the following form:

$$\begin{aligned}2w=&K_1\left[\cos\theta\cos(\psi-\phi)\left(\frac{d\theta}{d\rho}\right)+\frac{\sin\theta\cos(\psi-\phi)}{\rho}\left(\frac{d\psi}{d\phi}\right)\right]^2\\&+K_2\left[\sin(\psi-\phi)\left(\frac{d\theta}{d\rho}\right)+\frac{\sin\theta\cos\theta\sin(\psi-\phi)}{\rho}\left(\frac{d\psi}{d\phi}\right)\right]^2\\&+K_3\left[\sin^2\theta\cos^2(\psi-\phi)\left(\frac{d\theta}{d\rho}\right)^2\right.\\&\left.+\frac{\sin^4\theta\sin^2(\psi-\phi)}{\rho^2}\left(\frac{d\psi}{d\phi}\right)^2\right].\end{aligned} \tag{9}$$

The equation for ψ with the linear ansatz $\psi=N\phi+\alpha$ reduces to $\sin 2[(N-1)\phi+\alpha]=0$. Thus, in contrast to the case of equal elastic constant (7), only solutions with winding number $N=1$ and $\alpha=0$ or $\alpha=\pi/2$ (Fig. 1) are possible. Two ratios of the elastic constants span the phase space of independent control parameters for this functional. In this paper we use the ratios $\nu=K_1/K_2$ and $\kappa=K_3/K_2$. Minimization of the functional Eq. (9) yields the equilibrium values of the angle α and the corresponding profiles $\theta(\rho)$ for each point in the phase plane (ν,κ). An analysis of the phase diagram of all possible skyrmionic solutions for the functional Eq. (9) and of their stability limits must be done numerically and will be presented elsewhere (see, also, Ref. [19]). The nonlinear dependence of the energy density (9) on $\alpha=\psi-\phi$ will in the general case only admit solutions for certain fixed values of the parameter α. In some cases optimal values of α can be obtained without any calculation. For example, when $K_1=K_3$ the minimum of the functional Eq. (9) with respect to α does not depend on the distributions $\theta(\rho)$ and is reached either at $\alpha=0$ $(K_1<K_2)$ or $\alpha=\pi/2$ $(K_1>K_2)$. In this contribution we consider the skyrmions with $\alpha=\pi/2$ which are of a special interest. First, because these structures turned out to be stable for the relationship between the elastic constants $\nu>1$ which holds in all known nematics [16] (see the end of this section). Second, the equilibrium distribution of $\mathbf{n}(\mathbf{r})$ in this divergence-free structure can be described analytically. In this structure, the vector $\mathbf{n}$ rotates in the planes perpendicular to the radial directions in the skyrmion [Fig. 1(b)]. As follows from Eq. (9), for $\alpha=\pi/2$, the first term in the Frank energy equals zero, and the skyrmion is formed under the influence of twist and bend interactions only.

We demonstrate in detail this important and mathematically interesting case as a representative for the whole class of axisymmetric skyrmions in nematics. After integrating with respect to ϕ the functional Eq. (1) reduces to the following form:

$$\begin{aligned}W=&\pi K_2\int_0^\infty\omega\,d\rho\\=&\pi K_2\int_0^\infty\left[\left(\frac{d\theta}{d\rho}+\frac{\sin\theta\cos\theta}{\rho}\right)^2+\kappa\frac{\sin^4\theta}{\rho^2}\right]\rho\,d\rho,\end{aligned} \tag{10}$$

where ω is a reduced energy density per length, and the ratio $\kappa=K_3/K_2$ characterizes relative contributions of the twist and bend in the total energy balance. The Euler equation for this functional

$$\frac{d^2\theta}{d\rho^2}+\frac{1}{\rho}\frac{d\theta}{d\rho}-\frac{\sin 2\theta\cos 2\theta}{2\rho^2}-\kappa\frac{\sin^2\theta\sin 2\theta}{\rho^2}=0 \tag{11}$$

has the first integral

$$\left(\rho\frac{d\theta}{d\rho}\right)^2=\sin^2\theta(\cos^2\theta+\kappa\sin^2\theta)\equiv G(\theta). \tag{12}$$

Integration of Eq. (12) leads to the equation

$$\cot\theta+\sqrt{\cot^2\theta+\kappa}=\rho_0/\rho. \tag{13}$$

Here, $\rho_0>0$ is an integration constant. Using this equation we can write the solutions for the skyrmion in the following form:

$$\cot\theta=\frac{1-\kappa(\rho/\rho_0)^2}{2(\rho/\rho_0)},\quad \psi=\phi+\pi/2, \tag{14}$$

or in Cartesian coordinates

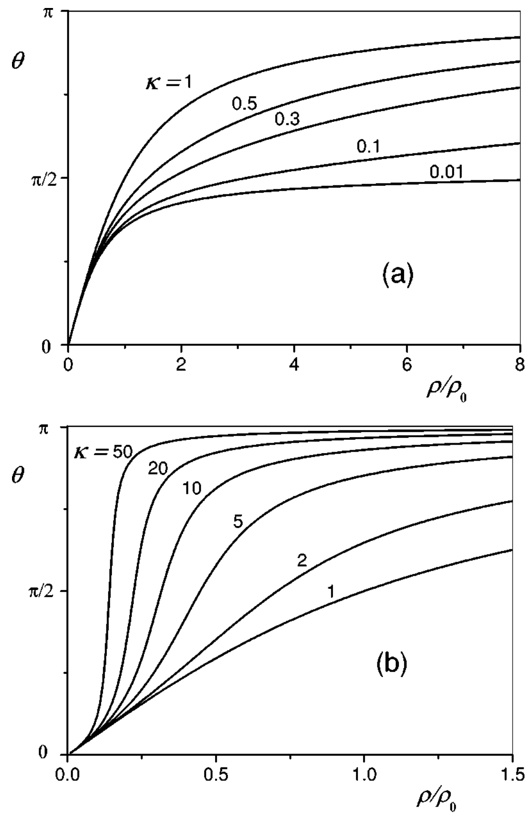

FIG. 2. Skyrmion profiles for different values of the parameter $\kappa = K_3/K_2$. For weak κ (a) the profiles have arrowlike shape; for large κ (b) the skyrmions have strongly localized cores separated from the outer part by a thin "domain wall."

$$\mathbf{n} = [-4\Omega(\rho/\rho_0)\sin\phi,\ \ 4\Omega(\rho/\rho_0)\cos\phi,$$
$$\Omega\sqrt{[1-\kappa(\rho/\rho_0)^2]^2 - 4(\rho/\rho_0)^2}], \tag{15}$$

where $\Omega = 1/\sqrt{[1-\kappa(\rho/\rho_0)^2]^2 + 4(\rho/\rho_0)^2}$. Skyrmion profiles (14) for different values of κ are plotted in Fig. 2. Note that the energy of these solutions remains constant under a transformation of the type

$$\tilde{\theta}(\rho) = \theta(\rho/\lambda), \quad \lambda > 0. \tag{16}$$

Phase *trajectories* or *orbits* of the solutions

$$\rho_0 \frac{d\theta}{d\rho} = \sqrt{\cos^2\theta + \kappa\sin^2\theta}(\cos\theta + \sqrt{\cos^2\theta + \kappa\sin^2\theta}), \tag{17}$$

calculated from Eqs. (12) and (13), are plotted in phase-space $(\theta, d\theta/d\rho)$ (Fig. 3). The orbits (17) start in the points $(0, 2/\rho_0)$ and end in the common point $(\pi, 0)$ [Fig. 3(a)]. As ρ_0 varies in the region $[0,\infty)$ the trajectories (17) cover the whole phase space. Thus, the set of profiles (14) comprises *all possible* solutions of the boundary value problem (11) with (6). This means that these equations admit only solutions with $\Delta\theta = \pi$ and do not have solutions with higher winding numbers for θ as found for skyrmions in other models (see, e.g., Ref. [20]). Near the center of the skyrmion

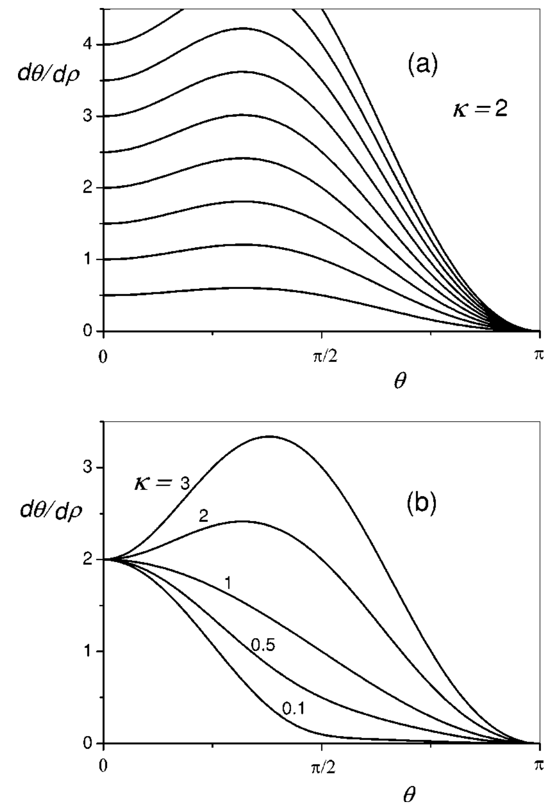

FIG. 3. Phase trajectories of the skyrmion solutions. (a) Phase space of the solutions for $\kappa = 2$; (b) phase trajectories for fixed ρ_0 (arbitrarily chosen as $\rho_0 = 1.0$) and different values of κ.

$(\rho \ll \rho_0)$ the angle θ varies linearly with the radius ($\theta = 2\rho/\rho_0$) and at large distances $(\rho \to \infty)$ $\theta = \pi - 2\rho_0/(\kappa\rho)$. These skyrmion structures are determined by a competition between twist and bend interactions [see Eq. (10)]: The former includes the derivative of θ, and, thus, suppresses strong variations of θ (Fig. 4). The latter favors states with small in-plane components. Evolution of the skyrmion profiles (Fig. 2) and the phase trajectories [Fig. 3(b)] under variation of the parameter κ reflect these tendencies. For weak *bending* energy (small κ) [see Fig. 4(a)] the profiles $\theta(\rho)$ have an arrowlike shape with the steepest slope at $\rho = 0$ [Fig. 2(a)]. In-plane states do not contribute to the twist energy. Thus, when the parameter κ approaches zero, the region with values of θ close to $\pi/2$ expands unlimitedly [Fig. 3(b)]. Finally, for $\kappa = 0$ it transforms into a nonlocalized structure (Fig. 5)

$$\theta = \arctan(2\rho/\rho_0). \tag{18}$$

Such configurations (vortices) occur in planar systems (e.g., in easy-plane ferromagnets) [21]. The energy of these vortices logarithmically increases with increasing vortex radius and becomes infinite in the case of nonlocalized magnetic vortices [21]. On the contrary, the nonlocalized skyrmion (18) has finite energy because, due to the absence of bend interactions $(\kappa = 0)$, in-plane inhomogeneous configurations yield no contribution to the skyrmion energy. With increasing values of κ the influence of bend distortions on the equi-

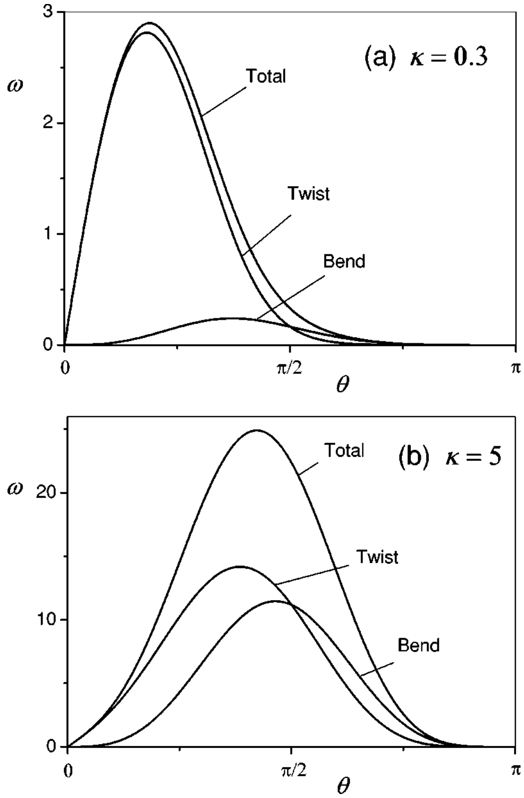

FIG. 4. Reduced energy density ω from Eq. (10) and twist and bend contributions as functions of angle θ for $\kappa=0.3$ (a) and $\kappa=5$ (b).

librium skyrmion structures increases [Fig. 4(b)]. The skyrmion profiles transform into structures with a strongly localized core separated from the outer parts by a thin "domain wall" [Fig. 2(b)]. For $\kappa>4/3$, the profiles have an inflection point at $\rho^*=\rho_0\sqrt{2\kappa^{-1}\sqrt{1-\kappa^{-1}}}$ [Fig. 3(b)]. The parameter ρ^* increases from zero at $\kappa=4/3$ to its maximum value at $\kappa^*=18(11-\sqrt{13})^{-1}=2.4343$; for larger κ parameter ρ^* monotonically decreases and approaches zero as κ tends to infinity. The Belavin-Polyakov skyrmion (7) with $\alpha=\pi/2$ is included in the general solution (14) as special case for $\kappa=1$.

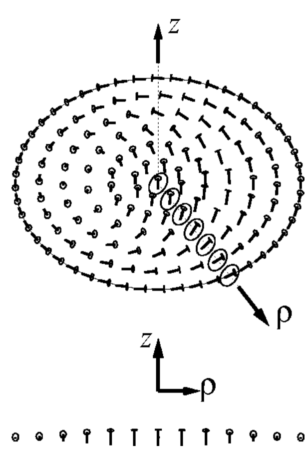

FIG. 5. Structure of **n** in the core of the nonlocalized skyrmion (17).

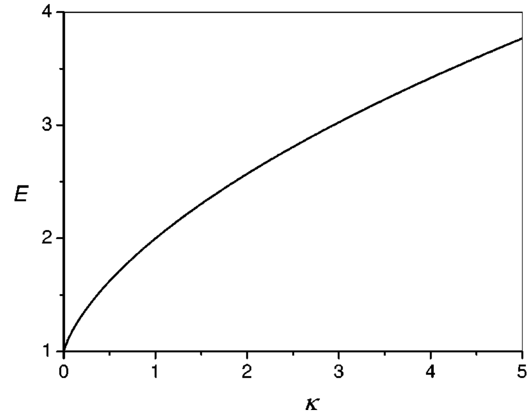

FIG. 6. Skyrmion energy (in units of $2\pi K_2$) as a function of the parameter κ.

Integration of the energy functional Eq. (10) with the solutions (14) yields the following expression for the skyrmion energy

$$E=2\pi K_2 J_1(\kappa)$$
$$=2\pi K_2\begin{cases}\left[1+\dfrac{\kappa}{2\sqrt{1-\kappa}}\ln\left(\dfrac{1+\sqrt{1-\kappa}}{1-\sqrt{1-\kappa}}\right)\right], & 0<\kappa<1\\ \left[1+\dfrac{\kappa}{\sqrt{\kappa-1}}\arcsin\sqrt{\dfrac{\kappa-1}{\kappa}}\right], & \kappa>1.\end{cases} \tag{19}$$

For zero bending, the skyrmion energy (19) has the minimum value $E=2\pi K_2$, and it increases monotonically with increasing κ (Fig. 6).

IV. STABILITY OF THE SOLUTIONS

In this section we investigate the stability of the solutions (14). Because the skyrmions have higher energy than the homogeneous state they can represent only local minima of the energy. In this case their *metastability* means that small perturbations of the structure (14) should only increase their energy. To check the stability one has to study the energy change under small perturbations of the localized structure (14). We exclude from consideration distortions that violate axial symmetry because they will only increase the skyrmion energy. Thus, we calculate the energy for a distorted structure

$$\tilde{\theta}(\rho)=\theta(\rho)+\xi(\rho),\quad \tilde{\psi}(\rho)=\psi(\phi)+\gamma,\quad \xi,\gamma\ll 1. \tag{20}$$

Distortions in the basal plane γ are assumed to be homogeneous because gradients of the angle ψ will also increase the energy. By expanding the energy of the distorted structure $\tilde{W}$ and keeping only terms up to second order in ξ and γ one obtains $\tilde{W}=W^{(0)}+W^{(2)}$, where $W^{(0)}=E$ (19) is the energy of the undisturbed skyrmion and the energy contribution $W^{(2)}$ includes terms quadratic with respect to the parameters ξ and γ

$$W^{(2)}=\pi K_2\int_0^\infty\frac{d\rho}{\rho}[w_\xi^{(2)}+w_\gamma^{(2)}],$$

$$w_\xi^{(2)}=\rho^2\left(\frac{d\xi}{d\rho}\right)^2+\sin^4\theta[\cot^4\theta-6\cot^2\theta+1+\kappa(3\cot^2\theta-1)]\xi^2,$$

$$w_\gamma^{(2)}=\left[\rho^2(\nu\cos^2\theta+\kappa\sin^2\theta-1)\left(\frac{d\theta}{d\rho}\right)^2+(\nu\sin^2\theta-\kappa\sin^4\theta-\sin^2\theta\cos^2\theta)\right]\gamma^2. \quad (21)$$

The angle $\theta(\rho)$ is given by Eq. (14). The term $w_\xi^{(2)}$ includes the energy of radial distortions. Perturbations $\xi(\rho)$ describe arbitrarily small deformations of the profile (14) which do not violate the boundary conditions (6), i.e., $\xi(0)=\xi(\infty)=0$. Stability of the skyrmion with respect to such deformations means that $W_\xi^{(2)}$ is positive for all functions $\xi(\rho)$ and can be established by solving the spectral problem for the functional $W_\xi^{(2)}(\xi,d\xi/d\rho)$ (see, e.g., Ref. [20]). In our case, however, there is an easier way to resolve the problem. By using the first integral Eq. (12) one can change the variable in the integral Eq. (21) as

$$W_\xi^{(2)}=\pi K_2\int_0^\infty\frac{w_\xi^{(2)}d\rho}{\rho}=\pi K_2\int_0^\pi\frac{d\theta}{\sqrt{G(\theta)}}\times\left[G(\theta)\left(\frac{d\xi}{d\theta}\right)^2+\frac{1}{2}\frac{d^2G(\theta)}{d\theta^2}\xi^2\right]. \quad (22)$$

Substitution of $\xi=\sqrt{G(\theta)}\zeta$ into Eq. (22) yields

$$W_\xi^{(2)}=\pi K_2\int_0^\pi G(\theta)\left(\frac{d\zeta}{d\theta}\right)^2d\theta. \quad (23)$$

Thus, for all distortions $\zeta(\theta)$ the energy contribution $W_\xi^{(2)}\geqslant 0$. Equality in this relation only holds for constant ζ. Under all other distortions $\zeta\neq\text{const}$, the skyrmion (14) preserves radial stability. In this connection we note that functions $\xi(\theta)=C\sqrt{G(\theta)}$ (C is an arbitrary constant) describe scaling transformations (16) for $\lambda=1+\xi$ and $|\xi|\ll 1$. This result is equivalent to the mentioned invariance of the solutions (14) under scaling (16). Integration in Eq. (21) yields the following result for the energy contributions due to in-plane distortions γ

$$W_\gamma^{(2)}=\pi K_2\int_0^\infty\frac{w_\gamma^{(2)}d\rho}{\rho}=A(\nu,\kappa)\gamma^2, \quad (24)$$

$$A(\nu,\kappa)=\pi K_2[\kappa(J_1-J_2)-2J_1+\nu(J_2+J_3)],$$

where the function $J_1(\kappa)$ is introduced in Eq. (19), and $J_2(\kappa)$ and $J_3(\kappa)$ are defined by

$$J_2=\begin{cases}\left[\dfrac{2-\kappa}{4(1-\kappa)}-\dfrac{\kappa^2}{8(1-\kappa)^{3/2}}\ln\left(\dfrac{1+\sqrt{1-\kappa}}{1-\sqrt{1-\kappa}}\right)\right], & 0<\kappa<1\\[2ex] \left[\dfrac{2-\kappa}{4(1-\kappa)}-\dfrac{\kappa^2}{8(1-\kappa)^{3/2}}\arcsin\sqrt{\dfrac{\kappa-1}{\kappa}}\right], & \kappa>1,\end{cases} \quad (25)$$

$$J_3=\begin{cases}\dfrac{1}{\sqrt{1-\kappa}}\ln\left(\dfrac{1+\sqrt{1-\kappa}}{1-\sqrt{1-\kappa}}\right), & 0<\kappa<1\\[2ex] \dfrac{2\arcsin\sqrt{\dfrac{\kappa-1}{\kappa}}}{\sqrt{\kappa-1}}, & \kappa>1.\end{cases} \quad (26)$$

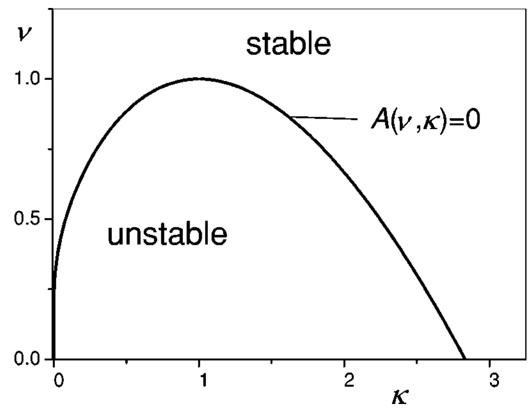

FIG. 7. Boundary of stability for the skyrmions (14) in the (ν,κ) phase plane. The structure is stable above the critical line $A(\nu,\kappa)=0$; cf. (24).

The skyrmion (14) is stable with respect to homogeneous in-plane distortions when $A(\nu,\kappa)>0$ (24). In the phase-plane (κ,ν) the critical line $A(\nu,\kappa)=0$ confines the region of skyrmion stability (Fig. 7). The area, where the skyrmion with $\alpha=\pi/2$ is unstable, is restricted to small values of κ and ν. The critical line approaches zero as $\kappa=4\exp(-2/\nu)$. In all known nematics the splay elastic constant is larger than the twist constant ($\nu>1$) [16]. In this practically important case the skyrmion is stable for arbitrary values of parameter κ.

V. DISCUSSION AND CONCLUSIONS

In this paper we have derived analytical solutions (14) for skyrmions minimizing the Frank free energy (1). The Frank functional includes in most general form invariants quadratic with respect to the spatial derivatives of the order parameter. And in this connection the functional Eq. (1) can be considered as a generalization of the O(3) nonlinear sigma model (4). In contrast to the Belavin-Polyakov solutions (7) which are degenerate with respect to the values of the phase α and may have arbitrary winding numbers N the nematic skyrmions, i.e., solutions (14) for the Frank energy have the topological charge $Q=1$. In addition, the angle α generally has a fixed value. Only in the special case of equal elastic constants the corresponding Belavin-Polyakov solutions exist for arbitrary values of α due to the isotropy of the elastic behavior (4). Here, we analyzed the special case where $\alpha = \pi/2$: the plane of rotation is perpendicular to the radial direction [Fig. 1(b)]. We come to the important result that, due to the anisotropy of the elastic behavior in the full expression of the Frank energy (1), the degeneracy of the skyrmionic solutions in the basal plane is lifted. However, the skyrmion energy (10) still preserves invariance under the scale transformations (16).

Invariance of the solutions (14) with respect to the scaling transformation (16) means that the Frank energy (1) for nematics does not stabilize skyrmions with well-defined sizes. The skyrmions (14) are in the state of *indifferent equilibrium* with respect to arbitrary compression ($0<\lambda<1$) or stretching ($\lambda>1$). Thus, they turn out to be unstable under the influence of applied electric or magnetic fields in conformance with the *Hobart-Derrick theorem* on localized solutions for this type of functionals [22]. To demonstrate that, we consider a nematic in a magnetic and electric field and calculate the variation of its energy [(1) with (2) contribution] under the scaling transformation (16). The new energy $\tilde{W}(\tilde{\theta})$ can be expressed as the following combination of the energy contributions (1) and (2) in $W(\theta)$:

$$\tilde{W}(\tilde{\theta}) = W_0 + W_f \lambda^2,$$

$$W_0 = \int_0^\infty w(\theta)\rho\, d\rho, \quad W_f = \int_0^\infty w_f(\theta)\rho\, d\rho. \qquad (27)$$

It is clear from Eq. (27) that the functional (1) with the energy contribution (2) does not have local minima for axisymmetric solutions $\theta(\rho)$. For any of these configurations, the energy (27) can decrease by unlimited compression (if $W_f>0$), or expansion ($W_f<0$) of the core. This result, known as *Hobart-Derrick theorem* [22], applies to any interaction functional consisting of only two types of terms: quadratic with respect to spatial derivatives (1) and those including no spatial derivatives (2). It is remarkable that this radial instability can be eliminated only by adding new terms that do not belong to the mentioned classes. Hence, skyrmions can be stabilized by terms with spatial derivatives to the power higher than two (e.g., fourth-order gradient terms in the Skyrme model [5,17]). In condensed-matter systems such fourth-order terms arise in models with complex competing interactions [23]. The skyrmions in such systems have been investigated in Ref. [24]. Another important mechanism stabilizing multidimensional localized structures is provided by specific interactions occurring in noncentrosymmetric systems due to chiral symmetry breaking [7,25]. Phenomenologically, these *chiral* interactions are described by invariants linear with respect to the first spatial derivatives (known as *Lifshitz invariants*) [20,25]. The stabilizing influence of the chiral interactions on multidimensional localized structures in noncentrosymmetric liquid crystals, i.e., cholesterics will be considered in a separate contribution [18].

There can be other axisymmetric and localized solutions for the model (1). As was mentioned above, in some cases the skyrmion with $\psi=\phi$ [Fig. 1(a)] has lower energy than the skyrmion (14). Skyrmions with more complex structures and interacting skyrmions may be an interesting topic for future investigations. We remind that the isotropic model (4) has multiskyrmion solutions [4], multitwisted structures [26], and skyrmion lattices [27].

The Belavin-Polyakov solutions [4] as nonlinear, topologically nontrivial, and localized objects have proved to be of great interest in the context of theoretical models in condensed-matter physics. A practical realization of such skyrmionic textures in real magnetic systems, however, is problematic due to the extreme idealization of the model (4) and the specific character of these solutions. Nematic liquid crystals should provide more favorable conditions for such experiments. The nematic skyrmions could be generated in the homeotropic texture similarly to isolated bubble domains in some cholesterics (e.g., by dynamic scattering [16,28]), and the relatively slow dynamics in liquid-crystal systems may facilitate the observation of topological excitations.

In this paper the analysis of nematic skyrmions has been restricted to straight infinite lines in unbounded ideal materials. In confined nematics these localized structures may create loops or end at defects or confining surfaces. The influence of anchoring effects and interactions with disclinations add new aspects to the physics of nematic skyrmions. The formation of so-called ‘‘thick threads’’ and their loops by dynamic scattering [29] may be considered as an indirect indication of the existence of the nematic skyrmions. String-like textures generated by rapid cooling through the isotropic-to-nematic phase transitions in regular [30] and polymer nematics [31] may be another indication for the existence of nematic skyrmions.

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